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SLOPE DEFLECTION EQUATIONS FOR CURVED MEMBERS

By Keith T. Fowler, Jun. ASCE

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PAPERS

SLOPE DEFLECTION EQUATIONS FOR CURVED MEMBERS

By Keith T. Fowler,[1] Jun. ASCE

Synopsis

The purpose of this paper is to illustrate a rapid method of determining the slope deflection equations for curved or haunched members through the use of the neutral point concept or the column analogy. A complete derivation and three examples are included. The equations obtained are valid for any deflection or rotation confined to a single plane. No restriction is placed on the applied loads as long as they lie in the plane.

Introduction

The determination of the slope deflection equations for curved or haunched members is complicated by the fact that each structure requires a particular equation—that is, no simple expression such as the equation for a straight member with a constant moment of inertia is obtainable. The nature of the applied forces has no effect on the form of the equations. Through the use of the neutral point concept or the column analogy it is possible to derive expressions which are perfectly general—the variables being computed in separate calculations.

The method of analysis is intended primarily for structures similar to those illustrated by Fig. 1. The limiting feature of this theory is that each elastic joint requires three simultaneous equations if it is free to deflect in two directions and to rotate.

Notation.—The following letter symbols are adopted for use in this paper and for the guidance of discussers:

d = deflection terms in Eqs. 41 (special case);
E = Young's modulus of elasticity;
H = horizontal reaction; H_{FE} = fixed-end reaction;
I = rectangular moment of inertia of structural member;

Note.—Written comments are invited for publication; the last discussion should be submitted by August 1, 1950.

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M = bending moment defined by appropriate subscript; M_{FE} = fixed-end moment; M_{AB} represents the moment reaction at joint A of member AB;

s = distances measured along the neutral axis of a member; ds = small increments of distance s;

U = bending strain energy produced by generally applied forces

$$= \int \frac{M^2\,ds}{2EI};$$

V = vertical reaction; V_{FE} = fixed-end reaction;

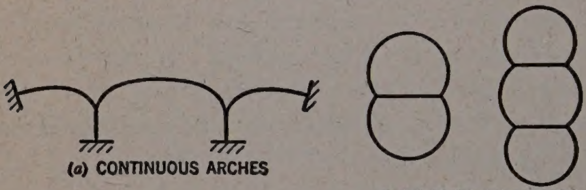
X = the x-translation of an abutment;

x = axis and abscissa distances about the neutral point or centroid of the analogous column;

Y = the y-translation of an abutment;

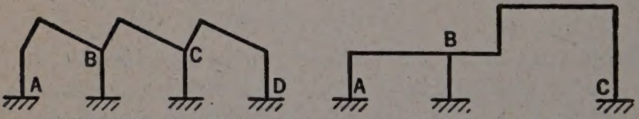
y = axis and ordinate distances about the neutral point or centroid of the analogous column; and

θ = rotation of an abutment.



(a) CONTINUOUS ARCHES

(b) AIRCRAFT BULKHEADS



(c) INDUSTRIAL BUILDING

(d) INDUSTRIAL BUILDING

FIG. 1.—TYPICAL STRUCTURES

The sign convention used is:

(1) Deflections, horizontally to the right, vertically upward, and rotationally clockwise with regard to the reference axes, are positive; and

(2) Force reactions, acting horizontally to the right and vertically upward, and clockwise moment reactions, are positive.

GENERAL DESCRIPTION

The basic principle involved in this method is the use of distortions of the neutral point or centroid of the analogous column in expressing the moments, thrusts, and shears in terms of the joint displacements. Fig. 2 illustrates a structure incorporating the possible joint deflections and rotations.

The procedure of analysis is as follows:

(1) Divide the structure into the least possible number of individual units and compute the fixed-end reactions due to the applied forces by any convenient method. In the structure illustrated by Fig. 1(c), members AB and CD would be treated as single units; the same procedure would be applied to members AB

and BC in Fig. 1(*d*). All reactions must be expressed in agreement with the sign convention.

(2) Assume a convenient direction for the reference axes. The fixed-end reactions must be parallel to the reference axes.

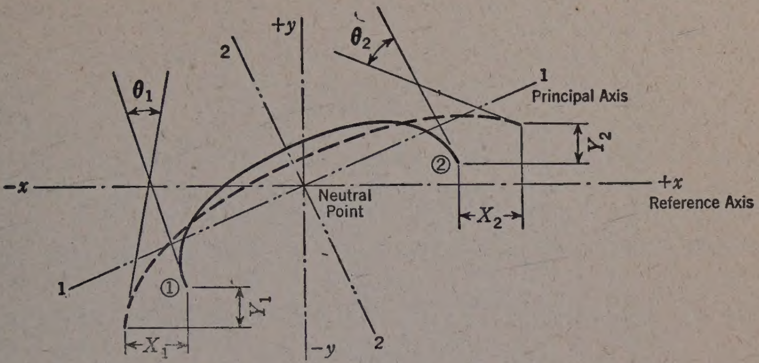


FIG. 2.—POSSIBLE JOINT DEFLECTIONS AND ROTATIONS

(3) Compute the following terms for the centroidal reference axes of each unit (all reference axes being parallel): (*a*) $\sum \frac{ds}{EI} = A$; (*b*) $\sum \left(y^2 \frac{ds}{EI} \right) = I_x$; (*c*) $\sum \left(x^2 \frac{ds}{EI} \right) = I_y$; and (*d*) $\sum \left(x\, y \frac{ds}{EI} \right) = I_{xy}$.

(4) Write the equations for the horizontal and vertical reactions, and the moment reactions at the ends of each member in terms of the fixed-end reactions and the unknown deflections and rotations.

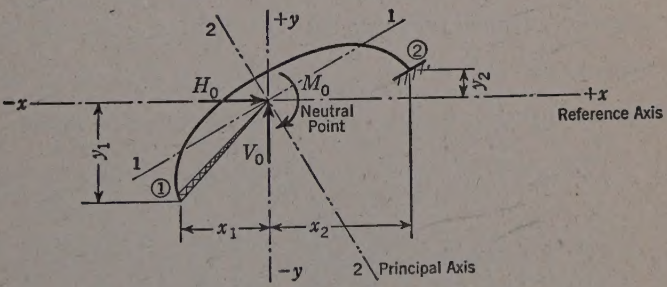


FIG. 3.—REACTIONS ON A DEFORMED STRUCTURE

(5) Let, at each joint, $\sum M = 0, \sum H = 0, \sum V = 0$.

(6) Solve the required number of simultaneous equations for the unknown deflections and rotations at the various joints and substitute the values into the equations for the reactions acting on each member as obtained in step (4).

The theoretical derivation of the equations involved in the foregoing steps, is arranged, for simplicity, in the Appendix. For example, equations for the reactions on a deformed structure, incorporating Eqs. 26 to 34 in the Appendix, are evolved as follows: Referring to Fig. 3—

$$M_1 = M_{FE} + M_0 - H_0\,y_1 + V_0\,x_1 \ldots\ldots\ldots\ldots (1a)$$

$$M_2 = M_{FE} - M_0 + H_0\,y_2 - V_0\,x_2 \ldots\ldots\ldots\ldots (1b)$$

$$H_1 = H_{FE} + H_0 \ldots\ldots\ldots\ldots (1c)$$

$$H_2 = H_{FE} - H_0 \ldots\ldots\ldots\ldots (1d)$$

$$V_1 = V_{FE} + V_0 \ldots\ldots\ldots\ldots (1e)$$

and

$$V_2 = V_{FE} - V_0 \ldots\ldots\ldots\ldots (1f)$$

The neutral point forces, in terms of the unknown deflections and rotations, are

$$V_0 = \Bigl\{(X_1 - X_2 - \theta_1\,y_1 + \theta_2\,y_2)\Bigl[\sum\Bigl(x\,y\,\frac{ds}{E\,I}\Bigr)\Bigr] + (Y_1 - Y_2 + \theta_1\,x_1 - \theta_2\,x_2)\Bigl[\sum\Bigl(y^2\,\frac{ds}{E\,I}\Bigr)$$

$$\div\Bigl\{\Bigl[\sum\Bigl(y^2\,\frac{ds}{E\,I}\Bigr)\Bigr]\Bigl[\sum\Bigl(x^2\,\frac{ds}{E\,I}\Bigr)\Bigr] - \Bigl[\sum\Bigl(x\,y\,\frac{ds}{E\,I}\Bigr)\Bigr]^2\Bigr\}\ldots(2$$

$$H_0 = \Bigl\{(X_1 - X_2 - \theta_1\,y_1 + \theta_2\,y_2)\Bigl[\sum\Bigl(x^2\,\frac{ds}{E\,I}\Bigr)\Bigr] + (Y_1 - Y_2 + \theta_1\,x_1 - \theta_2\,x_2)\Bigl[\sum\Bigl(x\,y\,\frac{ds}{E\,I}\Bigr)$$

$$\div\Bigl\{\Bigl[\sum\Bigl(y^2\,\frac{ds}{E\,I}\Bigr)\Bigr]\Bigl[\sum\Bigl(x^2\,\frac{ds}{E\,I}\Bigr)\Bigr] - \Bigl[\sum\Bigl(x\,y\,\frac{ds}{E\,I}\Bigr)\Bigr]^2\Bigr\}\ldots(2$$

and

$$M_0 = \frac{\theta_1 - \theta_2}{\sum \frac{ds}{E\,I}} \ldots\ldots\ldots\ldots (2c)$$

The inelastic bracket always extends from point 1, Fig. 3, to the neutral point. In cases where the principal axes coincide with the reference axes, Eqs. 2 are greatly simplified since the expression $\sum\Bigl(x\,y\,\frac{ds}{E\,I}\Bigr) = I_{xy}$ is equal to zero.

In the column analogy concept, Eqs. 2 take the following forms:

$$M_1 = M_{FE} - \Bigl\{\frac{W}{A} + \Bigl[\frac{M_x\,I_y - M_y\,I_{xy}}{I_x\,I_y - (I_{xy})^2}\Bigr]\,y_1 + \Bigl[\frac{M_y\,I_x - M_x\,I_{xy}}{I_x\,I_y - (I_{xy})^2}\Bigr]\,x_1\Bigr\} \ldots (3a)$$

$$M_2 = M_{FE} + \Bigl\{\frac{W}{A} + \Bigl[\frac{M_x\,I_y - M_y\,I_{xy}}{I_x\,I_y - (I_{xy})^2}\Bigr]\,y_2 + \Bigl[\frac{M_y\,I_x - M_x\,I_{xy}}{I_x\,I_y - (I_{xy})^2}\Bigr]\,x_2\Bigr\} \ldots (3b)$$

$$H_1 = H_{FE} + \Bigl[\frac{M_x\,I_y - M_y\,I_{xy}}{I_x\,I_y - (I_{xy})^2}\Bigr] \ldots\ldots\ldots\ldots (3c)$$

$$H_2 = H_{FE} - \Bigl[\frac{M_x\,I_y - M_y\,I_{xy}}{I_x\,I_y - (I_{xy})^2}\Bigr] \ldots\ldots\ldots\ldots (3d)$$

$$V_1 = V_{FE} - \Bigl[\frac{M_y\,I_x - M_x\,I_{xy}}{I_x\,I_y - (I_{xy})^2}\Bigr] \ldots\ldots\ldots\ldots (3e)$$

and

$$V_2 = V_{FE} + \left[\frac{M_y I_x - M_x I_{xy}}{I_x I_y - (I_{xy})^2} \right] \ldots\ldots\ldots\ldots (3f)$$

in which

$$W = \theta_2 - \theta_1 \ldots\ldots\ldots\ldots (4a)$$

$$M_x = X_1 - X_2 - \theta_1 y_1 + \theta_2 y_2 \ldots\ldots\ldots\ldots (4b)$$

and

$$M_y = Y_2 - Y_1 - \theta_1 x_1 + \theta_2 x_2 \ldots\ldots\ldots\ldots (4c)$$

The results from either the neutral point method or the column analogy method are identical. The neutral point method is used in the examples. No attempt is made to present examples which represent practicable structural designs or loadings. Three examples are included to illustrate the flexibility of the method and to show the treatment for special cases. The structures are assumed to be homogeneous, in which case E may be taken as equal to unity. The amounts of inertia of the various members are only relative.

EXAMPLE A

Example A is illustrated by Fig. 4, in which the three arches are identical. The calculation of the fixed-end reactions and of the constants for the structure is not a necessary part of this paper. Fig. 5 illustrates the dimensions of the arches, the centroid ordinates of the segment points being as follows:

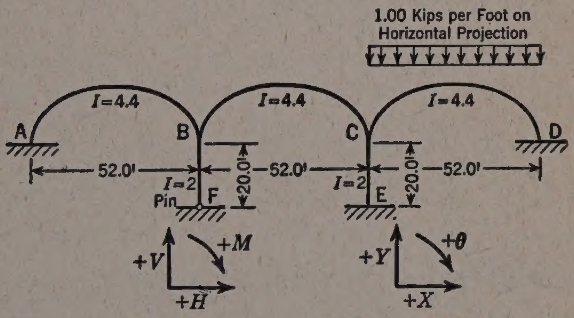


FIG. 4.—CONTINUOUS IDENTICAL ARCHES

Point	x	y
1	2.2	5.7
2	6.5	5.1
3	10.7	4.2
4	14.8	2.6
5	18.6	0.5
6	21.9	−2.3
7	24.4	−5.8
8	25.7	−10.0

In Example A, $E = 1$; $ds/I = 1$; and, by the formulas in step (3), $A = 16$; $I_x = 444.4$; $I_y = 4{,}922$; and $I_{xy} = 0$. Fig. 6 illustrates the fixed-end reactions for the arch CD (step (1)). Example A has four unknowns—X_B, X_C, θ_B, and θ_C. Vertical deflections do not enter the problem since member shortening has been neglected.

Arch AB.—Point A, Fig. 4, is designated as point 1. The values θ_A, Y_A, and Y_B are equal to zero. Substituting appropriate values in Eqs. 2,

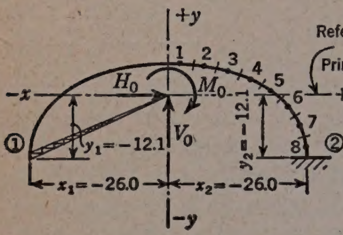


Fig. 5.—Arch Dimensions and Properties

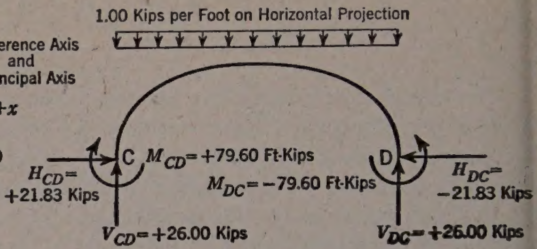

1.00 Kips per Foot on Horizontal Projection
C
D
$M_{CD} = +79.60$ Ft-Kips
$M_{DC} = -79.60$ Ft-Kips
$H_{CD} = +21.83$ Kips
$H_{DC} = -21.83$ Kips
$V_{CD} = +26.00$ Kips
$V_{DC} = +26.00$ Kips

Fig. 6.—Fixed-End Reactions on Arch CD

$$V_0 = -\frac{\theta_B x_B}{\Sigma\left(x^2 \frac{ds}{I}\right)} = -\frac{\theta_B (26.0)}{4{,}922.6} = -0.00529\,\theta_B \ldots\ldots\ldots (5a)$$

$$H_0 = \frac{-X_B + \theta_B y_B}{\Sigma\left(y^2 \frac{ds}{I}\right)} = \frac{-X_B + \theta_B(-12.1)}{444.4}$$

$$= -0.00225\,X_B - 0.0272\,\theta_B \ldots (5b)$$

and

$$M_0 = -\frac{\theta_B}{\Sigma \frac{ds}{I}} = -\frac{\theta_B}{16} = -0.0625\,\theta_B \ldots\ldots\ldots\ldots\ldots (5c)$$

Then, from Eqs. 1,

$$M_{AB} = M_0 - H_0(-12.1) + V_0(-26.0) = -0.2535\,\theta_B - 0.0272\,X_B \ldots (6a)$$

$$M_{BA} = -M_0 + H_0(-12.1) - V_0(26.0) = 0.5295\,\theta_B + 0.0272\,X_B \ldots\ldots (6b)$$

$$[H_{AB} = H_0 = -0.00225\,X_B - 0.0272\,\theta_B \ldots\ldots\ldots\ldots\ldots (6c)$$

$$H_{BA} = -H_0 = 0.00225\,X_B + 0.0272\,\theta_B \ldots\ldots\ldots\ldots\ldots (6d)$$

$$V_{AB} = V_0 = -0.00529\,\theta_B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6e)$$

and

$$V_{BA} = -V_0 = 0.00529\,\theta_B \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6f)$$

Arch BC.—Point B, Fig. 4, is designated as point 1. Only the vertical deflections Y_B and Y_C are equal to zero. From the same general equations used for arch AB,

$$V_0 = \frac{\theta_B x_B - \theta_C x_C}{\Sigma\left(x^2 \frac{ds}{I}\right)} = \frac{\theta_B(-26.0) - \theta_C(26.0)}{4{,}922.6}$$

$$= -0.00529\,\theta_B - 0.00529\,\theta_C \ldots (7a)$$

$$H_0 = \frac{X_B - X_C - \theta_B y_B + \theta_C y_C}{\Sigma\left(y^2 \frac{ds}{I}\right)} = \frac{X_B - X_C - \theta_B(-12.1) + \theta_C(-12.1)}{444.4}$$

$$= 0.00225\,X_B - 0.00225\,X_C + 0.0272\,\theta_B - 0.0272\,\theta_C \tag{7b}$$

and

$$M_0 = \frac{\theta_B - \theta_C}{\Sigma\frac{ds}{I}} = \frac{\theta_B - \theta_C}{16} = 0.0625\,\theta_B - 0.0625\,\theta_C \tag{7c}$$

From Eqs. 1,

$$M_{BC} = M_0 - H_0(-12.1) + V_0(-26.0)$$

$$= 0.5295\,\theta_B - 0.2535\,\theta_C + 0.0272\,X_B - 0.0272\,X_C \tag{8a}$$

$$M_{CB} = -M_0 + H_0(-12.1) - V_0(26.0)$$

$$= -0.2535\,\theta_B + 0.5295\,\theta_C - 0.0272\,X_B + 0.0272\,X_C \tag{8b}$$

$$H_{BC} = H_0 = 0.00225\,X_B - 0.00225\,X_C + 0.0272\,\theta_B - 0.0272\,\theta_C \tag{8c}$$

$$H_{CB} = -H_0 = -0.00225\,X_B + 0.00225\,X_C - 0.0272\,\theta_B + 0.0272\,\theta_C \tag{8d}$$

$$V_{BC} = V_0 = -0.00529\,\theta_B - 0.00529\,\theta_C \tag{8e}$$

and

$$V_{CB} = -V_0 = 0.00529\,\theta_B + 0.00529\,\theta_C \tag{8f}$$

Arch CD.—Arch CD, Fig. 4, contains fixed-end reactions which must be included in the equations. Referring to Fig. 6, Eqs. 2 (considering point C as point 1) yield

$$V_0 = \frac{\theta_C x_C}{\Sigma\left(x^2 \frac{ds}{I}\right)} = \frac{\theta_C(-26.0)}{4{,}922.6} = -0.00529\,\theta_C \tag{9a}$$

$$H_0 = \frac{X_C - \theta_C y_C}{\Sigma\left(y^2 \frac{ds}{I}\right)} = \frac{X_C - \theta_C(-12.1)}{444.4} = 0.00225\,X_C + 0.0272\,\theta_C \tag{9b}$$

and

$$M_0 = \frac{\theta_C}{\Sigma\frac{ds}{I}} = \frac{\theta_C}{16} = 0.0625\,\theta_C \tag{9c}$$

Hence, by Eqs. 1,

$$M_{CD} = M_{FE} + M_0 - H_0(-12.1) + V_0(-26.0)$$

$$= 79.60 + 0.5295\,\theta_C + 0.0272\,X_C \tag{10a}$$

$$M_{DC} = M_{FE} - M_0 + H_0(-12.1) - V_0(26.0)$$

$$= -79.60 - 0.2535\,\theta_C - 0.0272\,X_C \tag{10b}$$

$$H_{CD} = H_{FE} + H_0 = 21.83 + 0.00225\,X_C + 0.0272\,\theta_C \tag{10c}$$

$$H_{DC} = H_{FE} - H_0 = -21.83 - 0.00225\,X_C - 0.0272\,\theta_C \tag{10d}$$

$$V_{CD} = V_{FE} + V_0 = 26.00 - 0.00529\,\theta_C \ldots\ldots\ldots\ldots (10e)$$

and

$$V_{DC} = V_{FE} - V_0 = 26.00 + 0.00529\,\theta_C \ldots\ldots\ldots\ldots (10f)$$

Column BF.—The equations for a vertical column with one end pinned are illustrated by Fig. 16 in the Appendix. Therefore, from Eqs. 38, also in the Appendix,

$$M_{BF} = 3\left(\frac{2}{20}\right)\left(\theta_B - \frac{X_B}{20}\right) = 0.3000\,\theta_B - 0.0150\,X_B \ldots\ldots (11a)$$

$$H_{BF} = -\frac{M_{BF}}{20} = -0.0150\,\theta_B + 0.00075\,X_B \ldots\ldots\ldots\ldots (11b)$$

and

$$H_{FB} = -H_{BF} = 0.0150\,\theta_B - 0.00075\,X_B \ldots\ldots\ldots\ldots (11c)$$

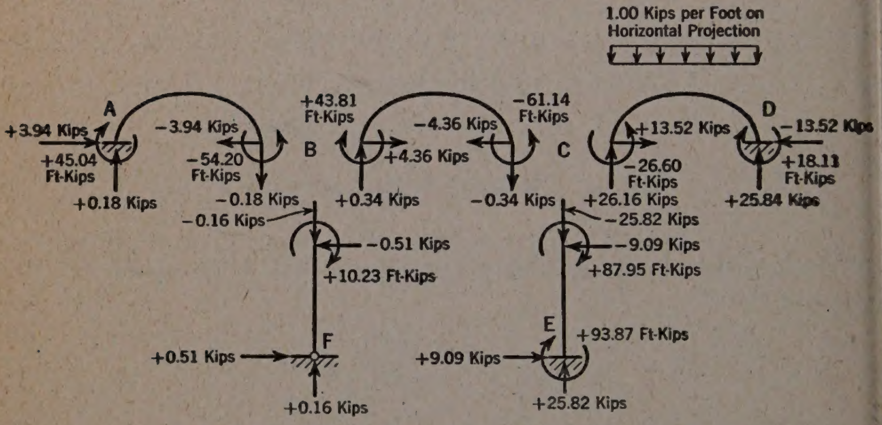


FIG. 7.—CONCLUDING STEPS, SOLUTION OF EXAMPLE A

Column CE.—The reactions in terms of the joint deflections and rotations for a vertical fixed-end column are expressed by Eqs. 36 in the Appendix (refer to Fig. 15):

$$M_{CE} = 2\left(\frac{2}{20}\right)\left(2\,\theta_C - \frac{3\,X_C}{20}\right) = 0.4000\,\theta_C - 0.0300\,X_C \ldots (12a)$$

$$M_{EC} = 2\left(\frac{2}{20}\right)\left(\theta_C - \frac{3\,X_C}{20}\right) = 0.2000\,\theta_C - 0.0300\,X_C \ldots\ldots (12b)$$

$$H_{CE} = -\frac{M_{CE} + M_{EC}}{20} = -0.0300\,\theta_C + 0.0030\,X_C \ldots\ldots (12c)$$

and

$$H_{EC} = -H_{CE} = 0.0300\,\theta_C - 0.0030\,X_C \ldots\ldots\ldots\ldots (12d)$$

Four unknowns are involved in the solution of Example A. The required simultaneous equations are as follows:

$$\sum M_B = 0 = 1.3590\,\theta_B - 0.2535\,\theta_C + 0.0394\,X_B - 0.0272\,X_C \quad (13a)$$

$$\sum M_C = 0 = -0.2535\,\theta_B + 1.4590\,\theta_C - 0.0272\,X_B + 0.0244\,X_C + 79.60 \quad (13b)$$

$$\sum H_B = 0 = 0.0394\,\theta_B - 0.0272\,\theta_C + 0.00525\,X_B - 0.00225\,X_C \quad (13c)$$

and

$$\sum H_C = 0 = -0.0272\,\theta_B + 0.0244\,\theta_C - 0.00225\,X_B + 0.0075\,X_C + 21.83 \quad (13d)$$

Solving Eqs. 13 simultaneously, $\theta_B = -33.25$ counterclockwise; $\theta_C = -29.62$ counterclockwise; $X_B = -1{,}345$ to the left; and $X_C = -3{,}328$ to the left. The results obtained by substituting the computed deflections and rotations into the equations for the various joint reactions are illustrated by Fig. 7.

EXAMPLE B

The structure illustrated by Fig. 8 is sometimes encountered in modern aircraft design. Before analyzing the problem it is necessary to fix the structure in space by drawing axes through one of the joints. At the fixed joint the deflections and rotation are equal to zero; and, in this example, $X_B = X_A = \theta_B = Y_B$. The linear shortening of beam AB can be neglected; so the solution to the example requires the determination of only θ_A and Y_A. Two simultaneous equations will suffice for any type of loading on the structure. The three-celled fuselage bulkhead illustrated by Fig. 1(b) would involve seven unknown quantities for any manner of loading.

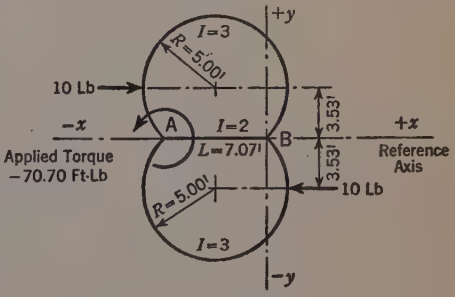


FIG. 8.—EXAMPLE B

Fig. 9 illustrates the fixed-end reactions (step (1)) and the properties of the individual units about their centroids. Assuming $E = 1$, the formulas in step (3) yield: $A = 7.87$; $I_x = 59.70$; and $I_y = 119.20$. Because the principal axes of each member are parallel to the reference axes, the term $\sum\left(x\,y\,\frac{ds}{I}\right)$ is zero in all cases.

Upper Shell AB.—From Eqs. 2, the neutral point forces, in terms of the joint rotations and deflections, are

$$V_o = \frac{Y_A + \theta_A\,x_A}{\sum\left(x^2\,\frac{ds}{I}\right)} = \frac{Y_A + \theta_A\,(-3.53)}{119.20} = 0.00839\,Y_A - 0.0296\,\theta_A \quad (14a)$$

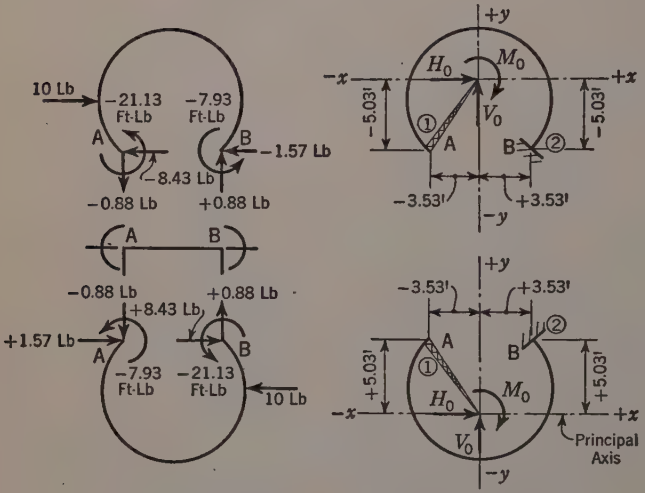
$$H_0 = \frac{-\theta_A\, y_A}{\Sigma\left(y^2 \frac{ds}{I}\right)} = \frac{-\theta_A\,(-5.03)}{59.70} = 0.08425\,\theta_A \ldots\ldots\ldots (14b)$$

and

$$M_0 = \frac{\theta_A}{\Sigma \frac{ds}{I}} = \frac{\theta_A}{7.87} = 0.1270\,\theta_A \ldots\ldots\ldots\ldots\ldots\ldots (14c)$$

Hence, from Eqs. 1,

$$\begin{aligned} M_{AB} &= -21.13 + M_0 - H_0\,(-5.03) + V_0\,(-3.53) \\ &= -21.13 - 0.0296\,Y_A + 0.6551\,\theta_A \ldots (15a) \end{aligned}$$



(a) FIXED-END REACTIONS (b) ELASTIC PROPERTIES

FIG. 9.—STEP 1, EXAMPLE B

$$\begin{aligned} M_{BA} &= -7.93 - M_0 + H_0\,(-5.03) - V_0\,(3.53) \\ &= -7.93 - 0.0296\,Y_A - 0.4465\,\theta_A \ldots (15b) \end{aligned}$$

$$H_{AB} = -8.43 + H_0 = -8.43 + 0.08425\,\theta_A \ldots\ldots\ldots (15c)$$

$$H_{BA} = -1.57 - H_0 = -1.57 - 0.08425\,\theta_A \ldots\ldots\ldots (15d)$$

$$V_{AB} = -0.88 + V_0 = -0.88 + 0.00839\,Y_A - 0.0296\,\theta_A \ldots (15e)$$

and

$$V_{BA} = +0.88 - V_0 = 0.88 - 0.00839\,Y_A + 0.0296\,\theta_A \ldots (15f)$$

Lower Shell AB.—The neutral point forces and moment, from Eqs. 4, are

$$V_0 = \frac{Y_A + \theta_A\, x_A}{\Sigma\left(x^2 \frac{ds}{I}\right)} = \frac{Y_A + \theta_A\,(-3.53)}{119.20} = 0.00839\,Y_A - 0.0296\,\theta_A \ldots (16a)$$

$$H_0 = \frac{-\theta_A y_A}{\Sigma\left(y^2 \frac{ds}{I}\right)} = \frac{-\theta_A (5.03)}{59.70} = -0.08425\,\theta_A \ldots\ldots\ldots (16b)$$

and

$$M_0 = \frac{\theta_A}{\Sigma \frac{ds}{I}} = \frac{\theta_A}{7.87} = 0.1270\,\theta_A \ldots\ldots\ldots\ldots\ldots (16c)$$

Then (by Eqs. 1),

$$M_{AB} = -7.93 + M_0 - H_0 (5.03) + V_0 (-3.53)$$
$$= -7.93 - 0.0296\,Y_A + 0.6551\,\theta_A \ldots (17a)$$

$$M_{BA} = -21.13 - M_0 + H_0 (5.03) - V_0 (3.53)$$
$$= -21.13 - 0.0296\,Y_A - 0.4465\,\theta_A \ldots (17b)$$

$$H_{AB} = 1.57 + H_0 = 1.57 - 0.08425\,\theta_A \ldots\ldots\ldots\ldots (17c)$$

$$H_{BA} = 8.43 - H_0 = 8.43 + 0.08425\,\theta_A \ldots\ldots\ldots\ldots (17d)$$

$$V_{AB} = -0.88 + V_0 = -0.88 + 0.00839\,Y_A - 0.0296\,\theta_A \ldots (17e)$$

and

$$V_{BA} = 0.88 - V_0 = 0.88 - 0.00839\,Y_A + 0.0296\,\theta_A \ldots\ldots\ldots (17f)$$

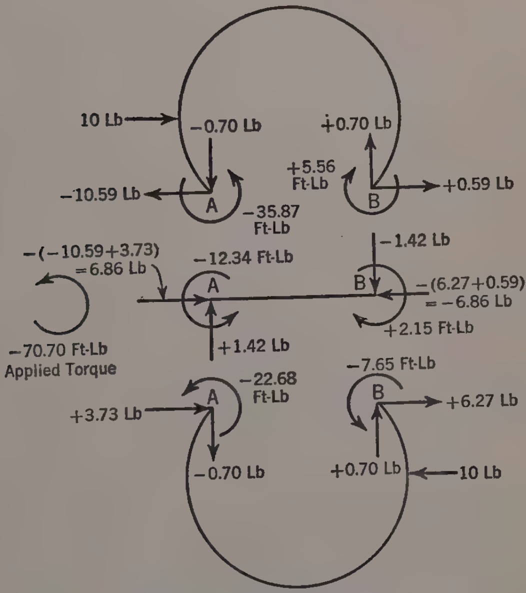


Fig. 10.—Concluding Steps, Solution of Example B

Beam AB.—Eqs. 35 (as illustrated by Fig. 15) supply the necessary relationships for the computation of

$$M_{AB} = 2\left(\frac{2}{7.07}\right)\left(2\,\theta_A - \frac{3\,Y_A}{7.07}\right) = 1.131\,\theta_A - 0.240\,Y_A \ldots (18a)$$

$$M_{BA} = 2\left(\frac{2}{7.07}\right)\left(\theta_A - \frac{3\,Y_A}{7.07}\right) = 0.566\,\theta_A - 0.240\,Y_A \qquad (18b)$$

$$V_{AB} = -\left(\frac{M_{AB} + M_{BA}}{7.07}\right) = -\,0.240\,\theta_A + 0.068\,Y_A \qquad (18c)$$

and

$$V_{BA} = -\,V_{AB} = 0.240\,\theta_A - 0.068\,Y_A \qquad (18d)$$

The conditions for equilibrium which supply the necessary simultaneous equations are: $\sum M_A = -\,70.70$ and $\sum V_A = 0$. Also, $\sum M_B = 0$; $\sum V_B = 0$; and $\sum H_A + \sum H_B = 0$. Hence,

$$\sum M_A = 0 = 2.4412\,\theta_A - 0.2992\,Y_A + 41.64 \qquad (19a)$$

and

$$\sum V_A = 0 = -\,0.2992\,\theta_A + 0.08478\,Y_A - 1.76 \qquad (19b)$$

Solving Eqs. 19 simultaneously $\theta_A = -\,25.59$ and $Y_A = -\,69.45$ ($X_A = X_B = \theta_B = Y_B = 0$).

Fig. 10 illustrates the reactions at the ends of the members produced by the applied forces.

EXAMPLE C

Fig. 11 illustrates Example C. The structure is conveniently subdivided into two frames and one column, as illustrated by Fig. 12(*a*). Hence, only the unknown values X_B and θ_B must be determined.

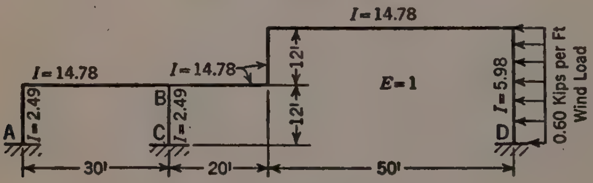


FIG. 11.—EXAMPLE C

Frame AB.—By the formulas in step (3): $A = 6.85$; $I_x = 109.22$; $I_y = 473.6$; and $I_{xy} = 128.55$. With these values and other properties in Fig. 12(*b*), Eqs. 2 yield:

$$V_0 = \frac{(-\,X_B + \theta_B\,y_B)I_{xy} + (-\,\theta_B\,x_B)I_x}{I_x\,I_y - (I_{xy})^2}$$

$$= \frac{[-\,X_B + \theta_B\,(4.23)]\,(128.55) - \theta_B\,(25.56)\,(109.22)}{(109.22)\,(473.6) - (128.55)^2}$$

$$= -\,0.0637\,\theta_B - 0.00364\,X_B \qquad (20a)$$

$$H_0 = \frac{(-\,X_B + \theta_B\,y_B)\,I_y + (-\,\theta_B\,x_B)\,I_{xy}}{I_x\,I_y - (I_{xy})^2}$$

$$= \frac{[-\,X_B + \theta_B\,(4.23)]\,(473.6) - \theta_B\,(25.56)\,(128.55)}{(109.22)\,(473.6) - (128.55)^2}$$

$$= -\,0.0363\,\theta_B - 0.01342\,X_B \qquad (20b)$$

and

$$M_0 = \frac{-\theta_B}{A} = \frac{-\theta_B}{6.85} = -0.146\,\theta_B \ldots\ldots\ldots\ldots (20c)$$

Therefore,

$$M_{AB} = M_0 - H_0(-7.77) + V_0(-4.44) = -0.145\,\theta_B - 0.0881\,X_B \,.. (21a)$$

$$M_{BA} = -M_0 + H_0(4.23) - V_0(25.56) = 1.6204\,\theta_B + 0.0362\,X_B \ldots (21b)$$

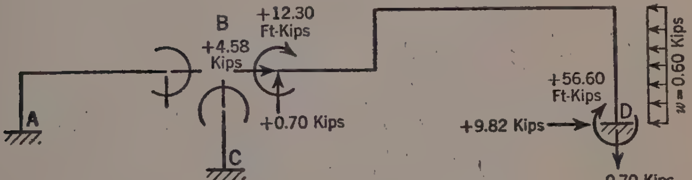
$$H_{AB} = H_0 = -0.0363\,\theta_B - 0.01342\,X_B \ldots\ldots\ldots (21c)$$

$$H_{BA} = -H_0 = 0.0363\,\theta_B + 0.01342\,X_B \ldots\ldots\ldots (21d)$$

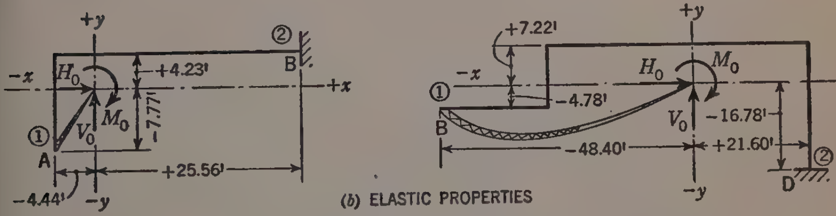
$$V_{AB} = V_0 = -0.0637\,\theta_B - 0.00364\,X_B \ldots\ldots\ldots (21e)$$

and

$$V_{BA} = -V_0 = 0.0637\,\theta_B + 0.00364\,X_B \ldots\ldots\ldots (21f)$$



(a) FIXED-END REACTIONS



(b) ELASTIC PROPERTIES

FIG. 12.—STEP 1, EXAMPLE C

Column BC.—From Eqs. 36 in the Appendix (refer to Fig. 15),

$$M_{CB} = 2\left(\frac{2.49}{12}\right)\left(\theta_B - \frac{3}{12}X_B\right) = 0.415\,\theta_B - 0.1038\,X_B \ldots (22a)$$

$$M_{BC} = 2\left(\frac{2.49}{12}\right)\left(2\,\theta_B - \frac{3}{12}X_B\right) = 0.830\,\theta_B - 0.1038\,X_B \ldots (22b)$$

and

$$H_{BC} = -H_{CB} = -\left(\frac{M_{CB} + M_{BC}}{12}\right) = -0.1038\,\theta_B + 0.0173\,X_B \,.. (22c)$$

Frame BD.—By Eqs. 2, the neutral point reactions in terms of the unknown joint rotation and deflection are

$$V_0 = \frac{[X_B - \theta_B(-4.78)](-277.6) + \theta_B(-48.40)(501.77)}{(501.77)(5{,}320.3) - (-277.6)^2}$$

$$= -0.0099\,\theta_B - 0.0001075\,X_B \,.. (23a)$$

$$H_0 = \frac{[X_B - \theta_B(-4.78)](5{,}320.3) + \theta_B(-48.40)(-277.6)}{(501.77)(5{,}320.3) - (-277.6)^2}$$

$$= 0.01505\,\theta_B + 0.00206\,X_B \ldots (23b)$$

and

$$M_0 = \frac{\theta_B}{9.56} = 0.1045\,\theta_B \ldots\ldots\ldots\ldots\ldots\ldots (23c)$$

Hence, by Eqs. 1,

$$M_{BD} = 12.3 + M_0 - H_0(-4.78) + V_0(-48.40)$$

$$= 12.3 + 0.6555\,\theta_B + 0.01506\,X_B \ldots (24a)$$

$$M_{DB} = 56.6 - M_0 + H_0(-16.78) - V_0(21.60)$$

$$= 56.6 - 0.1427\,\theta_B - 0.0322\,X_B \ldots (24b)$$

$$H_{BD} = 4.58 + H_0 = 4.58 + 0.01505\,\theta_B + 0.00206\,X_B \ldots\ldots\ldots (24c)$$

$$H_{DB} = 9.82 - H_0 = 9.82 - 0.01505\,\theta_B - 0.00206\,X_B \ldots\ldots\ldots (24d)$$

$$V_{BD} = 0.70 + V_0 = 0.70 - 0.0099\,\theta_B - 0.0001075\,X_B \ldots\ldots\ldots (24e)$$

and

$$V_{DB} = -0.70 - V_0 = -0.70 + 0.0099\,\theta_B + 0.0001075\,X_B \ldots (24f)$$

The required simultaneous equations are

$$\sum M_B = 12.3 + 3.1059\,\theta_B - 0.0525\,X_B = 0 \ldots\ldots\ldots\ldots (25a)$$

and

$$\sum H_B = 4.58 - 0.0524\,\theta_B + 0.03278\,X_B = 0 \ldots\ldots\ldots\ldots (25b)$$

Therefore, $\theta_B = -6.50$ and $X_B = -149.5$. Fig. 13 illustrates the final reactions on the component parts of the structure.

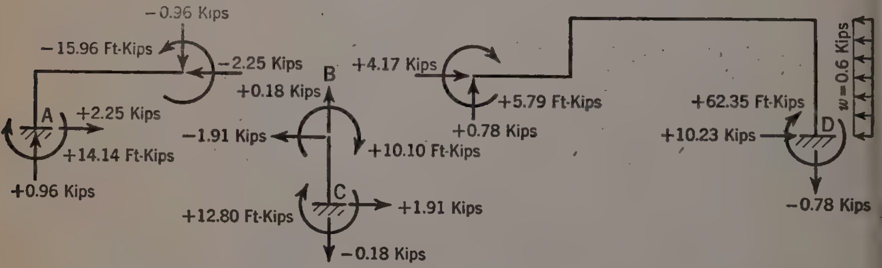


Fig. 13.—Concluding Steps, Solution of Example C

Conclusion

A method has been presented which is relatively convenient for analyzing continuous frames incorporating curved members. The extent of its application is dependent on the ability of the engineer to solve multiple simultaneous equations. Considerable accuracy is required if large numbers of equations are involved. (The method of iteration, which is in reality a successive approximation procedure, is particularly convenient for solving multiple simultaneous equations rapidly.)

Some of the features of the procedure may be summarized as follows:

1. No restriction is placed on the applied loading as long as it lies in the plane of the structure.
2. The equations automatically produce results which indicate the correct directions for the reactions. This feature is particularly convenient when using the equations to determine moment distribution constants.
3. Several loading conditions may be analyzed without altering the form of the equations—the fixed-end reactions being treated as separate steps.
4. When the structure is homogeneous, relative values of the moments of inertia and a modulus of elasticity equal to unity will suffice for all calculations.
5. As with the usual slope deflection solutions, the results are self-checking.

ACKNOWLEDGMENT

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APPENDIX

The neutral point concept and the column analogy have been presented in several forms in engineering literature. Few of the available references on the subject discuss the effects of distorting the position of the neutral point, which is of prime importance in this paper. Deflections and rotations at the joints of a structure produce a considerable change in the reactions. This Appendix is concerned with the values added to or subtracted from the fixed-end reactions of a structure by changes in the relative position of the supports.

Let point 1, Fig. 14(*a*) move parallel to the x–x reference axis a distance X_0 relative to point 2. Deflections to the right are positive. No joint rotation or vertical deflection is permitted. Then, from Fig. 14(*a*),

$$X_0 = X_1 - X_2 \qquad (26)$$

Fig. 14(*b*) illustrates the case in which point 1 deflects a vertical distance Y_0 relative to point 2, joint rotations and horizontal deflections being prevented. Considering deflections in the vertically upward direction as positive,

$$Y_0 = Y_1 - Y_2 \qquad (27)$$

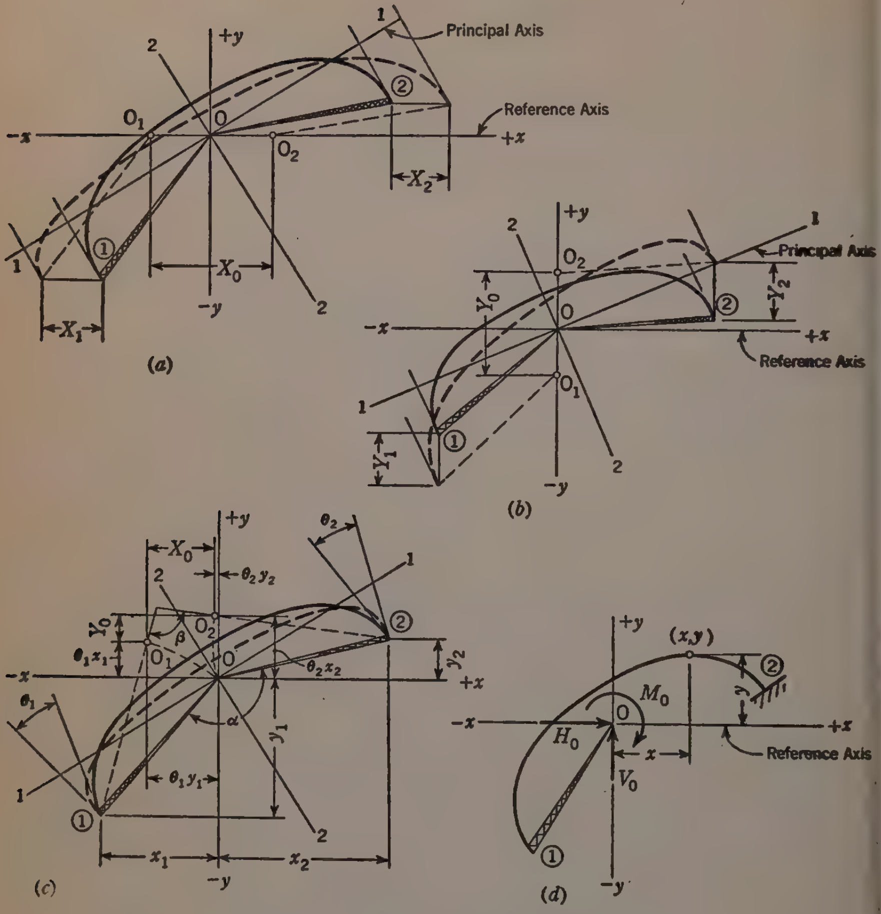
The final group of separate deflection studies is illustrated by Fig. 14(*c*), in which point 1 is rotated through the angle θ_0 relative to point 2. Clockwise rotations are positive. From Fig. 14(*c*),

$$\theta_0 = \alpha - \beta = \theta_1 - \theta_2 \qquad (28a)$$

$$X_0 = -\theta_1 y_1 + \theta_2 y_2 \quad \text{(28b)}$$

and

$$Y_0 = \theta_1 x_1 - \theta_2 x_2 \quad \text{(28c)}$$



(a) (b) (c) (d)

FIG. 14.—DERIVATION OF SLOPE DEFLECTION EQUATIONS

By combining Eqs. 26 through 28, the following general expressions are obtained:

$$\theta_0 = \theta_1 - \theta_2 \quad \text{(29a)}$$

$$X_0 = X_1 - X_2 - \theta_1 y_1 + \theta_2 y_2 \quad \text{(29b)}$$

and

$$Y_0 = Y_1 - Y_2 + \theta_1 x_1 - \theta_2 x_2 \quad \text{(29c)}$$

If the joints of a structure are not rigidly held in position while the structure is subjected to loading, the axial and shear strain energies can be neglected without incorporating a significant error. The bending strain energy produced by gradually applied forces is expressed as U and equals

$$U = \int \frac{M^2\, ds}{2\, E\, I} \qquad (30)$$

in which M is the bending moment at any point in the structure. Fig. 14(d) shows that the moment at any point (x,y) is equal to

$$M = M_0 + V_0\, x - H_0\, y \qquad (31)$$

The neutral point forces are positive as shown. Hence,

$$U = \int_1^2 \frac{(M_0 + V_0\, x - H_0\, y)^2\, ds}{2\, E\, I} \qquad (32)$$

It then follows that

$$X_0 = \frac{\partial U}{\partial H_0} = \int_1^2 \frac{(M_0 + V_0\, x - H_0\, y)\,(-\, y)\, ds}{E\, I}$$

$$= -\, V_0 \sum \left(x\, y \frac{ds}{E\, I} \right) + H_0 \sum \left(y^2 \frac{ds}{E\, I} \right)$$

$$= -\, V_0\, I_{xy} + H_0\, I_x \qquad (33a)$$

$$Y_0 = \frac{\partial U}{\partial V_0} = \int_1^2 \frac{(M_0 + V_0\, x - H_0\, y)\,(x)\, ds}{E\, I}$$

$$= V_0 \sum \left(x^2 \frac{ds}{E\, I} \right) - H_0 \sum \left(x\, y \frac{ds}{E\, I} \right) = V_0\, I_y - H_0\, I_{xy} \qquad (33b)$$

and

$$\theta_0 = \frac{\partial U}{\partial M_0} = \int_1^2 \frac{(M_0 + V_0\, x - H_0\, y)\,(1)\, ds}{E\, I} = M_0 \sum \frac{ds}{E\, I} = M_0\, A \qquad (33c)$$

About any pair of centroidal axes the terms $\sum \left(y \frac{ds}{E\, I} \right)$ and $\sum \left(x \frac{ds}{E\, I} \right)$ are equal to zero. In cases where the reference axes coincide with the principal axes, the term $\sum \left(x\, y \frac{ds}{E\, I} \right)$ is also equal to zero.

Solving Eq. 33a and Eq. 33b simultaneously,

$$V_0 = \frac{X_0\, I_{xy} + Y_0\, I_x}{I_x\, I_y - (I_{xy})^2} \qquad (34a)$$

and

$$H_0 = \frac{X_0\, I_y + Y_0\, I_{xy}}{I_x\, I_y - (I_{xy})^2} \qquad (34b)$$

Also, from Eq. 33c,

$$M_0 = \frac{\theta_0}{A} \qquad (34c)$$

Eqs. 1 through 4 in the paper incorporate Eqs. 34 into expressions for the reactions on a deformed structure. The general equations reduce to simple formulas when applied to a straight member with a constant moment of inertia if the member is parallel to one of the reference axes. In Fig. 15(a) a horizontal straight member that was initially fixed at the ends,

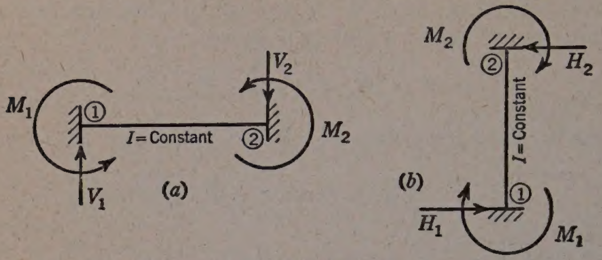


Fig. 15.—Straight Member, Initially Fixed at the Ends

$$M_1 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 - \frac{3\,Y_1}{L} + \frac{3\,Y_2}{L}\right) \qquad (35a)$$

$$M_2 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_2 + \theta_1 - \frac{3\,Y_1}{L} + \frac{3\,Y_2}{L}\right) \qquad (35b)$$

$$V_1 = V_{FE} - \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \qquad (35c)$$

and

$$V_2 = V_{FE} + \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \qquad (35d)$$

In Fig. 15(b), a vertical straight member that was initially fixed at the ends,

$$M_1 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 + \frac{3\,X_1}{L} - \frac{3\,X_2}{L}\right) \qquad (36a)$$

$$M_2 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_2 + \theta_1 + \frac{3\,X_1}{L} - \frac{3\,X_2}{L}\right) \qquad (36b)$$

$$H_1 = H_{FE} + \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \qquad (36c)$$

and

$$H_2 = H_{FE} - \left(\frac{M_1 + M_2 - \sum M_{FE}}{L}\right) \qquad (36d)$$

In Fig. 16(a), a vertical straight member with its upper end pinned,

$$M_1 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_1 + \frac{X_1}{L}\right) \qquad (37a)$$

$$H_1 = H_{FE} + \left(\frac{M_1 - M_{FE}}{L}\right);\qquad(37b)$$

and

$$H_2 = H_{FE} - \left(\frac{M_1 - M_{FE}}{L}\right).\qquad(37c)$$

In Fig. 16(*b*), a vertical straight member with its lower end pinned,

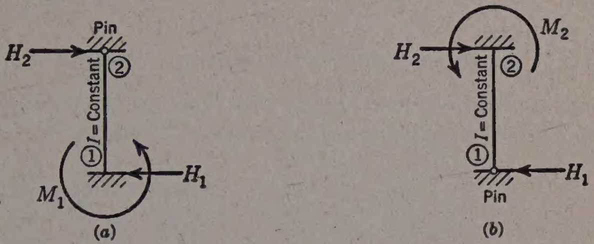


Fig. 16.—Equations for Vertical Straight Members with One End Pinned

$$M_2 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_2 - \frac{X_2}{L}\right);\qquad(38a)$$

$$H_2 = H_{FE} - \left(\frac{M_2 - M_{FE}}{L}\right)\qquad(38b)$$

and

$$H_1 = H_{FE} + \left(\frac{M_2 - M_{FE}}{L}\right).\qquad(38c)$$

In Fig. 17(*a*), a horizontal straight member with its left end pinned,

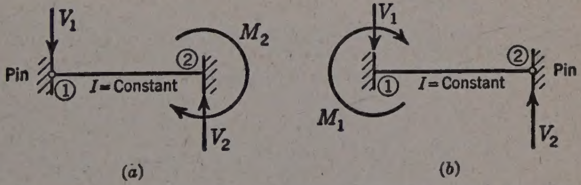


Fig. 17.—Equations for Horizontal Straight Members with One End Pinned

$$M_2 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_2 + \frac{Y_2}{L}\right)\qquad(39a)$$

$$V_2 = V_{FE} + \left(\frac{M_2 - M_{FE}}{L}\right)\qquad(39b)$$

and

$$V_1 = V_{FE} - \left(\frac{M_2 - M_{FE}}{L}\right)\qquad(39c)$$

and, in Fig. 17(*b*), a horizontal straight member with its right end pinned,

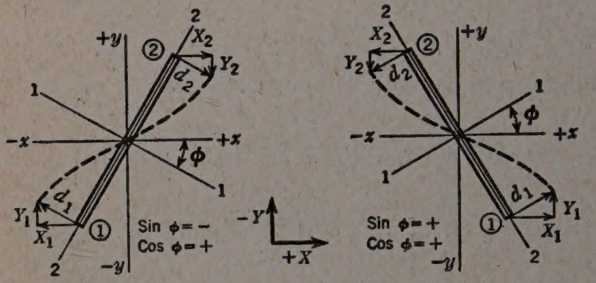
$$M_1 = M_{FE} + 3\,E\left(\frac{I}{L}\right)\left(\theta_1 - \frac{Y_1}{L}\right)\qquad(40a)$$

$$V_1 = V_{FE} - \left(\frac{M_1 - M_{FE}}{L}\right) \ldots\ldots\ldots\ldots\ldots (40b)$$

and

$$V_2 = V_{FE} + \left(\frac{M_1 - M_{FE}}{L}\right) \ldots\ldots\ldots\ldots\ldots (40c)$$

For an inclined straight member, the expression, $\sum\left(y^2 \frac{ds}{EI}\right)\sum\left(x^2 \frac{ds}{EI}\right) - \left[\sum\left(x\,y \frac{ds}{EI}\right)\right]^2$, is often a very small quantity. The equations illustrated in Fig. 15 can be modified to provide the equations for the reactions on such a member. Fig. 18 illustrates two possible cases. The angle ϕ is measured from the horizontal reference axis to the major principal axis. The clockwise direction is positive. Hence,



(a) SIN φ = − AND COS φ = + (b) SIN φ = + AND COS φ = +

Fig. 18.—Inclined Straight Members

$$X = d\cos\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41a)$$

$$Y = d\sin\phi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (41b)$$

and

$$d = \frac{X}{\cos\phi} = \frac{Y}{\sin\phi} \ldots\ldots\ldots\ldots\ldots\ldots (41c)$$

The deflection terms d_1 and d_2 replace the terms Y_1, Y_2, X_1, and X_2 in Eqs. 35 and 36. For example, Eq. 36a can be modified as follows:

$$M_1 = M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 + \frac{3\,d_1}{L} - \frac{3\,d_2}{L}\right)$$

$$= M_{FE} + 2\,E\left(\frac{I}{L}\right)\left(2\,\theta_1 + \theta_2 + \frac{3\,X_1}{L\cos\phi} - \frac{3\,X_2}{L\cos\phi}\right) \ldots (42)$$